DAMTP/97-18

QUANTUM HOMOGENEOUS SPACES AND COALGEBRA BUNDLES

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Abstract

It is shown that quantum homogeneous spaces of a quantum group H can be viewed as fibres of quantum fibrations with the total space H that are dual to coalgebra bundles. As concrete examples of such structures the fibrations with the quantum 2-sphere and the quantum hyperboloid fibres are considered.

1. INTRODUCTION

Among the variety of quantum spaces quantum homogeneous spaces seem to play a special role. Their symmetry structure is rich enough to display various properties of quantum group actions. Recently by studying the structure of quantum homogeneous spaces [3] we were lead to the notion of a coalgebra principal bundle [7] which generalises the notion of a quantum principal bundle [6] [10] [15] or a Hopf-Galois extension [11] [14]. A coalgebra principal bundle is built with a coalgebra C which plays the role of the 'structure group' (fibre) and two algebras M and P, the first of which plays the role of the base manifold and the second corresponds to the total space. The spaces C, M, P satisfy some conditions which in algebraic terms mean that P is an extension of M by C, known as a C-Galois extension (see Section 2 for definition). Since all the spaces involved in the definition of coalgebra principal bundles are either algebras or coalgebras and the notions of a coalgebra and an algebra are dual to each other, it seems natural to consider an object dual to a coalgebra principal bundle. This is built with an algebra A which plays the role of a fibre and the coalgebras $\overline{C}$ and C, the first of which is a base and the second is a total space. Again, in algebraic terms C is an extension of the coalgebra $\overline{C}$ by an algebra A which can be termed an *A-Galois coextension* (see Section 3 for definition). In this paper we show that the structure of quantum homogeneous spaces provides natural examples of such A-Galois coextensions.

Quantum homogeneous spaces M of Hopf algebras H that we study in Section 4 have the following remarkable property. As an algebra H is a C-Galois extension of M by a certain coalgebra C, while as a coalgebra it is an A-Galois coextension of (the same) C by M[1]. In geometric terms these quantum homogeneous spaces play the role of a base manifold in the former case and a fibre in the latter (the same total space in both cases). This suggests the notion of a CA-Galois biextension, which is very much reminiscent of bicrossproducts of [13], and we think might be well-worth studying.

Throughout the paper, all vector spaces are over a field k of characteristic 0, although the results can be extended to more general fields or even commutative rings. By an algebra we mean an associative algebra over k with unit denoted by 1. In a coalgebra C, Δ is the coproduct and $\epsilon : C \to k$ is the counit. We use the Sweedler notation for a coproduct, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation understood), for any $c \in C$.

2. COALGEBRA BUNDLES AND EMBEDDABLE H-SPACES

A coalgebra principal bundle is defined as follows. Let C be a coalgebra, P be an algebra and a right C-comodule. It means that there is a linear map $\Delta_P : P \to P \otimes C$, such that $(id \otimes \Delta) \circ \Delta_P = (\Delta_P \otimes id) \circ \Delta_P$ and $(id \otimes \epsilon) \circ \Delta_P = id$. The map Δ_P is known as a *right coaction*. Let $e \in C$ be group-like, i.e. $\Delta e = e \otimes e$, and let $M = P_e^{coC} = \{x \in P | \Delta_P(x) = x \otimes e\}$ be a space of coinvariants. Furthermore assume that the coaction is left-linear over M, i.e. $\Delta_P(xu) = x\Delta_P(u)$ for all $x \in M$, $u \in P$ and that $\Delta_P(1) = 1 \otimes e$. Then M is an algebra and the canonical map

$$\chi_M : P \otimes_M P \to P \otimes C, \qquad \chi_M : u \otimes v \mapsto u\Delta_P(v) \tag{2.1}$$

is well defined. We say that P is a *C-coalgebra principal bundle over M* or a *C-Galois extension of M* if the map χ_M is a bijection[2]. We denote this bundle by $P(M, C, e)$. This definition reduces to the definition of a quantum principal bundle with universal differential calculus of [6] if C is a Hopf algebra, Δ_P is an algebra map and $e = 1 \in C$ (cf. [4, Lemma 3.2] [12, Proposition 1.6]).

The notion of a coalgebra bundle is motivated by the structure of left quantum homogeneous spaces and, indeed, this is the context in which C-Galois extensions ap-

[1] We would like to stress, however, that from the point of view of entwining structures involved in the definition of coalgebra bundles in [7] those two bundles differ substantially. In the first case the spaces C and H while in the second H and M are entwined.

[2] In [7] a coalgebra principal bundle was defined in the framework of entwining structures. It is shown in [8] that the above definition is equivalent to the one in [7]. The same remark applies to the definition of a dual coalgebra bundle in Section 3.

peared for the first time in [20] (although studied for entirely different reasons there). Let H be a Hopf algebra and M be a left (resp. right) H-comodule algebra. It means that there is a linear map ${}_M\Delta : M \to H \otimes M$ such that $(id \otimes {}_M\Delta) \circ {}_M\Delta = (\Delta \otimes id) \circ {}_M\Delta$ and $(\epsilon \otimes id) \circ {}_M\Delta = id$, i.e. a left coaction (resp. a right coaction $\Delta_M : M \to M \otimes H$) which, furthermore, is an algebra map. Following [16] we say that M is a *left (*resp. *right) embeddable quantum homogeneous space* or simply a *left (*resp. *right) embeddable H-space* if there exists an algebra inclusion $i : M \hookrightarrow H$ such that $\Delta \circ i = (id \otimes i) \circ {}_M\Delta$ (resp. $\Delta \circ i = (i \otimes id) \circ \Delta_M$). When M is viewed as a subalgebra of H via i then the coaction coincides with the coproduct in H.

Given a left embeddable H-space M one defines the right ideal $J_R = i(M)^+ H$, where $i(M)^+ = \ker \epsilon \cap i(M)$. This ideal is a coideal of H, i.e. $\Delta(J_R) \subset H \otimes J_R \oplus J_R \otimes H$ and, obviously, $\epsilon(J_R) = 0$. Therefore $C_R = H/J_R$ is a coalgebra and a canonical surjection $\pi_R : H \to C_R = H/J_R$ is a coalgebra map. Furthermore, C_R is a right H-module with the action $\rho_R(c, h) = \pi_R(gh)$ for any $h \in H$, $c \in C_R$ and $g \in \pi_R^{-1}(c)$. Finally H is a right C-comodule with the coaction $\Delta_H = (id \otimes \pi_R) \circ \Delta$. If one chooses $e = \pi_R(1)$ then $H_e^{coC_R}$ is a subalgebra of H and Δ_H is left-linear over $H_e^{coC_R}$. Also, the canonical map $\chi_{H_e^{coC_R}}$ is a bijection [20, Lemma 1.3] or [3, Example 4.4]. Therefore H is a C-principal coalgebra bundle over $H_e^{coC_R}$. Furthermore, $i(M) \subset H_e^{coC_R}$, i.e. M is a subalgebra of $H_e^{coC_R}$. If M is isomorphic to $H_e^{coC_R}$ via i then clearly H is a coalgebra bundle over M. For example, the map $i : M \hookrightarrow H_e^{coC_R}$ is an isomorphism, if H is faithfully flat right or left M-module [20, Lemma 1.3] (cf. [2, Section I.3] for the definition and discussion of faithful flatness).

3. DUAL COALGEBRA BUNDLES AND EMBEDDABLE H-SPACES

First recall the definition of a dual coalgebra bundle [7] which generalises the object analysed in [19, Theorem II]. Let A be an algebra, let C be a coalgebra and a right A-module with the action $\rho_C : C \otimes A \to C$ and let $\kappa : A \to k$ be an algebra character. Define a vector space $J_\kappa = \rho_C(C, \ker \kappa)$ and a quotient space $\overline{C} = C/J_\kappa$. Let $\pi_\kappa : C \to \overline{C}$ be a canonical surjection. It can be easily shown [8] that if $\epsilon \circ \rho_C = \epsilon \otimes \kappa$ and

$$\Delta_\kappa \circ \rho_C = (id \otimes \rho_C) \circ (\Delta_\kappa \otimes id), \tag{3.1}$$

where $\Delta_\kappa = (\pi_\kappa \otimes id) \circ \Delta$, then J_κ is a coideal and therefore $\overline{C}$ is a coalgebra. Since $\overline{C}$ is a coalgebra and π_κ is a coalgebra map, C is a left $\overline{C}$ comodule with the coaction Δ_κ.

Since, moreover, C is a right A-module, (3.1) is equivalent to the right-linearity of the coaction Δ_κ over A. Next, one defines the cotensor product by

$$C\Box_{\overline{C}}C = \{b \otimes c \in C \otimes C | b_{(1)} \otimes \pi_\kappa(b_{(2)}) \otimes c = b \otimes \pi_\kappa(c_{(1)}) \otimes c_{(2)}\}.$$

Define a canonical map

$$\chi^{\overline{C}} : C \otimes A \to C\Box_{\overline{C}}C, \qquad \chi^{\overline{C}} : c \otimes a \mapsto c_{(1)} \otimes \rho_C(c_{(2)}, a). \tag{3.2}$$

We say that C is a *dual coalgebra A-principal bundle over $\overline{C}$* or an *A-Galois coextension of $\overline{C}$* if the canonical map $\chi^{\overline{C}}$ is a bijection. This bundle is denoted by $C(\overline{C}, A, \kappa)$.

We show that quantum embeddable homogeneous spaces provide examples of dual coalgebra bundles. Consider a *right* H-embeddable space M with embedding $i : M \hookrightarrow H$. Similarly as for left spaces, M can be interpreted as a quantum quotient space, i.e. as coinvariants of the coaction of certain coalgebra. Consider a left ideal $J_L = Hi(M)^+$. This ideal is a coideal of H, therefore $C_L = H/J_L$ is a coalgebra and a canonical surjection $\pi_L : H \to C_L$ is a coalgebra map. C_L is a left H-module with the action $\rho_L(h, c) = \pi_L(hg)$ for any $h \in H$, $c \in C_L$ and $g \in \pi_L^{-1}(c)$. Finally H is a left C-comodule with the coaction ${}_H\Delta = (\pi_L \otimes id) \circ \Delta$. If one chooses $e = \pi_L(1)$ then the space of coinvariants ${}^{coC_L}H_e = \{x \in H | \, {}_H\Delta(x) = e \otimes x\}$ is a subalgebra of H and, in nice cases such as the faithfully flat one, M is isomorphic to ${}^{coC_L}H_e$ by i. From now on we assume that $i(M) = {}^{coC_L}H_e$. M acts on H from the right via the multiplication, i.e. $\rho_H(h, x) = hi(x)$, thus J_L has the structure of J_κ above with $\kappa = \epsilon \circ i$. Clearly $\epsilon \circ \rho_H = \epsilon \otimes \kappa$. Moreover, ${}_H\Delta$ is right-linear over M, hence (3.1) holds. Thus we are in the setting needed for a dual coalgebra bundle over C_L, with a fibre M and the total space H. It remains to be checked that χ^{C_L} is a bijection. Consider a map

$$\tilde{\chi} : H\Box_{C_L}H \to H \otimes M, \qquad \tilde{\chi} : h \otimes g \mapsto h_{(1)} \otimes i^{-1}((Sh_{(2)})g),$$

where $S : H \to H$ is the antipode of H. First it needs to be shown that $\tilde{\chi}$ is well-defined. Take any $h \otimes g \in H\Box_{C_L}H$ and apply $id \otimes {}_H\Delta$ to $h_{(1)} \otimes (Sh_{(2)})g$,

$$\begin{aligned} h_{(1)} \otimes \pi_L((Sh_{(3)})g_{(1)}) \otimes (Sh_{(2)})g_{(2)} &= h_{(1)} \otimes \rho_L(Sh_{(3)}, \pi_L(g_{(1)})) \otimes (Sh_{(2)})g_{(2)} \\ &= h_{(1)} \otimes \rho_L(Sh_{(3)}, \pi_L(h_{(4)})) \otimes (Sh_{(2)})g \\ &= h_{(1)} \otimes e \otimes (Sh_{(2)})g. \end{aligned}$$

The second equality follows from the fact that $h \otimes g \in H\Box_{C_L}H$. Therefore $h_{(1)} \otimes (Sh_{(2)})g \in H \otimes {}^{coC_L}H_e$ and $\tilde{\chi}$ is well-defined. An elementary calculation shows that $\tilde{\chi}$ is an inverse of χ^{C_L}. Consequently there is a dual coalgebra bundle $H(C_L, M, \epsilon \circ i)$.

It is perhaps worth mentioning that every left homogeneous H-space M is a right H^{cop}-space, where H^{cop} is isomorphic to H as an algebra but has an opposite coproduct. If H has the invertible antipode then H^{cop} is also a Hopf algebra. Viewing a left embeddable H-space M as a right H^{cop}-space one can thus construct the bundle $H^{cop}(C_L^{cop}, M, \epsilon \circ i)$. Therefore M plays a double role: firstly it is a base for $H(M, C_R, \pi_R(1))$ constructed in Section 2 and secondly it is a fibre of $H^{cop}(C_L^{cop}, M, \epsilon \circ i)$.

4. TWO EXPLICIT EXAMPLES

The examples below are selected from a long list of well-known quantum homogeneous spaces such as quantum planes, homogeneous spaces of $E_\kappa(2)$ and $E_\kappa(3)$ etc, to all of which the above construction applies too. The first example is a non-trivial quantum spherical fibration of $SU_q(2)$ while the second one is a trivial quantum hyperbolic fibration of $E_q(2)$ (the word 'trivial' here means that the total space is a certain crossed product of a base and a fibre see [7] and [5] for details and examples).

4.1. Quantum Spherical Fibration of $SU_q(2)$

The quantum two-sphere [17] $S_q^2(\mu,\nu)$ is a polynomial complex algebra generated by the unit and x, y, z, and the relations

$$xz = q^2 zx, \qquad xy = -q(\mu - z)(\nu + z), \quad yz = q^{-2} zy, \qquad yx = -q(\mu - q^{-2}z)(\nu + q^{-2}z),$$

where μ, ν and $q \neq 0$ are real parameters, $\mu\nu \geq 0$, $(\mu,\nu) \neq (0,0)$. The quantum sphere is a $*$-algebra with the $*$-structure $x^* = -qy$, $z^* = z$.

The quantum sphere $S_q^2(\mu,\nu)$ is an $SU_q(2)$ homogeneous quantum space. $SU_q(2)$ is a complex algebra generated by the matrix of generators $\mathbf{t} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and the relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\delta = \delta\alpha + (q - q^{-1})\beta\gamma,$$

$$\beta\gamma = \gamma\beta, \quad \beta\delta = \delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \alpha\delta - q\beta\gamma = 1.$$

$SU_q(2)$ is a Hopf algebra of a matrix group type, i.e.

$$\Delta\mathbf{t} = \mathbf{t} \otimes \mathbf{t}, \quad \epsilon\mathbf{t} = 1, \quad S\mathbf{t} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix},$$

and has a $*$-structure given by $\delta = \alpha^*$, $\gamma = -q^{-1}\beta^*$. $S_q^2(\mu,\nu)$ is not only a quantum homogeneous space but also it is an embeddable $SU_q(2)$-space. Furthermore it is a left and right embeddable $SU_q(2)$-space [16] with the $*$-algebra inclusions

$$i_L(x) = \sqrt{\mu\nu}(q\alpha^2 - \beta^2) + (\mu - \nu)\alpha\beta, \quad i_L(y) = \sqrt{\mu\nu}(q\gamma^2 - \delta^2) + (\mu - \nu)\gamma\delta,$$

$$i_L(z) = -\sqrt{\mu\nu}(q\alpha\gamma - \beta\delta) - q(\mu-\nu)\beta\gamma,$$

$$i_R(x) = q\sqrt{\mu\nu}(\alpha^2 - q\gamma^2) - q(\mu-\nu)\alpha\gamma, \quad i_R(y) = \sqrt{\mu\nu}(q^{-1}\beta^2 - \delta^2) - q^{-1}(\mu-\nu)\beta\delta,$$

$$i_R(z) = \sqrt{\mu\nu}(\alpha\beta - q\gamma\delta) - q(\mu-\nu)\beta\gamma.$$

To simplify the forthcoming analysis assume that $\mu \neq \nu$. Then $S_q^2(\mu,\nu)$ depends on two real parameters q and $p = \frac{\sqrt{\mu\nu}}{\mu-\nu}$. Applying the analysis of Section 2 to $S_q^2(\mu,\nu)$ and using the map i_L above one defines a right ideal J_R in $SU_q(2)$ generated by

$$p(q\alpha^2 - \beta^2) + \alpha\beta - pq, \quad p(q\gamma^2 - \delta^2) + \gamma\delta + p, \quad p(q\alpha\gamma - \beta\delta) + q\beta\gamma.$$

and constructs the coalgebra $C_R = SU_q(2)/J_R$. C_R is spanned by $\pi_R(\alpha^n)$, $\pi_R(\delta^n)$, $n = 0, 1, 2, \ldots$ It is shown in [3] that the space of coinvariants $M = SU_q(2)_e^{coC_R}$ is isomorphic to $S_q^2(\mu,\nu)$. Therefore a coalgebra principal bundle $SU_q(2)(S_q^2(\mu,\nu), C_R, \pi_R(1))$ over $S_q^2(\mu,\nu)$ with fibre C_R is constructed.

On the other hand we can apply the analysis of Section 3 and use the map i_R which allows us to view $S_q^2(\mu,\nu)$ as a right embeddable $SU_q(2)$-space. We proceed to define a left ideal J_L in $SU_q(2)$ generated by

$$p(q\delta^2 - \beta^2) + \beta\delta - pq, \quad p(q\gamma^2 - \alpha^2) + \alpha\gamma + p, \quad p(\alpha\beta - q\gamma\delta) - q\beta\gamma.$$

and construct the coalgebra $C_L = SU_q(2)/J_L$. C_L is spanned by $\pi_L(\alpha^n)$, $\pi_L(\delta^n)$, $n = 0, 1, 2, \ldots$. Following [3] one deduces that the space of left coinvariants $M = {}^{coC_L}SU_q(2)_e$ is isomorphic to $S_q^2(\mu,\nu)$ and there is a dual coalgebra bundle $SU_q(2)(C_L, S_q^2(\mu,\nu), \kappa)$ over C_L with fibre $S_q^2(\mu,\nu)$. Here κ is a $*$-algebra character of $S_q^2(\mu,\nu)$ given by $\kappa(x) = q\sqrt{\mu\nu}$, $\kappa(y) = -\sqrt{\mu\nu}$ and $\kappa(z) = 0$.

Remarkably, it can be shown that although C_L and C_R are different as $SU_q(2)$ modules they are isomorphic to each other as coalgebras. The isomorphism is $\pi_L(\alpha^n) \mapsto \pi_R(\delta^n)$ and $\pi_L(\delta^n) \mapsto \pi_R(\alpha^n)$. Thus we identify C_L with C_R as coalgebras and conclude that $SU_q(2)$ can be viewed as a CA-Galois biextension: it is a C-Galois extension of $S_q^2(\mu,\nu)$ by C_R and it is an A-Galois coextension of C_R by $S_q^2(\mu,\nu)$.

4.2. Quantum Hyperbolic Fibration of $E_q(2)$

As another example take the quantum hyperboloid X_q [18]. X_q is a complex algebra generated by z_+, z_- and the identity, and the relation

$$z_+z_- = q^2 z_- z_+ + (1-q^2),$$

where q is a real number. X_q can also be viewed as a deformation of the complex plane obtained by the stereographic projection of $S^2_q(\mu,0)$ [9]. X_q is a $*$-algebra with involution $z_+^* = z_-$ and it is a quantum homogeneous space of $E_q(2)$. The latter is a complex algebra generated by v, v^{-1}, n_+, n_- subject to the following relations [21] [22]

$$vn_\pm = q^2 n_\pm v, \qquad n_+n_- = q^2 n_- n_+ \quad vv^{-1} = v^{-1}v = 1.$$

$E_q(2)$ is a quantum group with a coproduct, counit and the antipode:

$$\Delta v^{\pm 1} = v^{\pm 1}\otimes v^{\pm 1}, \quad \Delta n_\pm = n_\pm \otimes 1 + v^{\pm 1}\otimes n_\pm,$$

$$\epsilon(v^{\pm 1}) = 1, \quad \epsilon(n_\pm) = 0, \quad S(v^{\pm 1}) = v^{\mp 1}, \quad S(n_\pm) = -v^{\mp 1}n_\pm.$$

$E_q(2)$ is a $*$-algebra, $v^* = v^{-1}$, $n_+^* = n_-$. It is shown in [1] that X_q is a left embeddable $E_q(2)$-space with the coaction ${}_{X_q}\Delta(z_\pm) = n_\pm \otimes 1 + v^{\pm 1}\otimes z_\pm$ and the $*$-algebra embedding $i_L(z_\pm) = v^{\pm 1} + n_\pm$. It can be easily seen that X_q is also a right embeddable $E_q(2)$-space with the coaction $\Delta_{X_q}(z_\pm) = z_\pm \otimes v^{\pm 1} + q^{\mp 1}\otimes v^{\pm 1}n_\mp$ and the $*$-algebra embedding $i_R(z_\pm) = v^{\pm 1} + q^{\mp 1}v^{\pm 1}n_\mp$. In both cases X_q comes from the construction described in Sections 2 and 3 with the right ideal J_R generated by $v^{\pm 1} + n_\pm - 1$ and the left ideal J_L generated by $v^{\pm 1} - q^{\pm 1}n_\pm - 1$. The corresponding coalgebras $C_R = E_q(2)/J_R$ and $C_L = E_q(2)/J_L$ are isomorphic to each other as coalgebras (but, of course, differ by their $E_q(2)$- module structure). C_R is spanned by group-like $\pi_R(v^n)$, $n \in \mathbf{Z}$ and C_L is spanned by group-like $\pi_L(v^n)$, $n \in \mathbf{Z}$, and the isomorphism is given by $\pi_R(v^n) \mapsto \pi_L(v^n)$. Thus we can identify C_L, C_R with the coalgebra C spanned by elements c_p, $p \in \mathbf{Z}$, such that $\Delta c_p = c_p \otimes c_p$. The right and left actions of $E_q(2)$ on C are given by

$$\rho_R(c_p, v^k) = c_{p+k}, \quad \rho_R(c_p, n_\pm) = q^{2p}(c_p - c_{p\pm 1}),$$

$$\rho_L(v^k, c_p) = c_{p+k}, \quad \rho_L(n_\pm, c_p) = q^{2p\mp 1}(c_{p\pm 1} - c_p).$$

Therefore the quantum Euclidean group $E_q(2)$ is a CA-Galois biextension built on C and X_q. In other words we have a coalgebra and a dual coalgebra principal bundles, $E_q(2)(X_q, C, c_0)$ over X_q with fibre C and $E_q(2)(C, X_q, \kappa)$ over C with fibre X_q. The map κ is a $*$-character of X_q given by $\kappa(z_\pm) = 1$.

To obtain a slightly more topological insight into the structure of $E_q(2)$ one equips C with an algebra structure of $\mathbf{C}[Z, Z^{-1}]$ by $c_n \mapsto Z^n$ and thus turns it into a Hopf algebra. By requiring that π_L and π_R be $*$-maps the compatible $*$-structure is obtained on $\mathbf{C}[Z, Z^{-1}]$ as $Z^* = Z^{-1}$. Thus $C \equiv \mathbf{C}[Z, Z^{-1}]$ is an algebra of functions on a (classical) circle S^1. $E_q(2)$ can be viewed then as the S^1-bundle over X_q as well as a dual bundle over a circle with quantum hyperbolic fibres.

ACKNOWLEDGEMENT

Research supported by the EPSRC grant GR/K02244.

References

[1] F. Bonechi *et.al.*, *Commun. Math. Phys.* 175, 161, (1996).

[2] N. Bourbaki. "Commutative Algebra", Addison-Wesley, Reading, (1972).

[3] T. Brzeziński, *J. Math. Phys.* 37, 2388, (1996).

[4] T. Brzeziński, *J. Geom. Phys.* 20, 349, (1996).

[5] T. Brzeziński, Preprints q-alg/9603016 and q-alg/9612013, (1996)

[6] T. Brzeziński and S. Majid. *Commun. Math. Phys.*, 157, 591, (1993). Erratum 167, 235, (1995).

[7] T. Brzeziński and S. Majid. Preprint DAMTP/95-74, q-alg/9602022.

[8] T. Brzeziński and P.M. Hajac. *In preparation.*

[9] C.-S. Chu, P.-M. Ho and B. Zumino. *Z. Phys. C*, 70, 339, (1996).

[10] M. Durdević. *Commun. Math. Phys.*, 175, 457, (1996) and Preprint q-alg/9412005, (1994)

[11] H.F. Kreimer and M. Takeuchi. *Indiana Univ. Math. J.* 30, 675, (1981).

[12] P.M. Hajac. *Commun. Math. Phys.* 182, 579 (1996).

[13] S. Majid. *J. Algebra*, 130, 17, (1990).

[14] S. Montgomery. "Hopf Algebras and Their Actions on Rings". CBMS 82, AMS, Providence, 1993.

[15] M. Pflaum. *Commun. Math. Phys.* 166, 279, (1994).

[16] P. Podleś. *Commun. Math. Phys.*, 170, 1, (1995).

[17] P. Podleś. *Lett. Math. Phys.*, 14, 193, (1987).

[18] K. Schmüdgen. *Commun. Math. Phys.*, 159, 217, (1994).

[19] H.-J. Schneider. *Israel J. Math.*, 72, 167, (1990).

[20] H.-J. Schneider. *J. Algebra*, 152, 289, (1992).

[21] L.L. Vaksman and L.I. Korogodskii. *Soviet Math. Dokl.*, 39, 173, (1990).

[22] S.L. Woronowicz. *Commun. Math. Phys.*, 136, 399, (1991).